IEP-TPI Tool Technical Appendix

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The TPI tool methodology presented here aims at calculating the ”perpetual annuity” which represents the ”fundamental pricing” (i.e. the annual perpetual cost for servicing a debt as interest rate) associated with an issuance of a MS's public debt, given its assignment to a credit rating class. Particularly, this annuity prices the credit-standing migration risk to which a given credit rating class is exposed, and it is based on a theoretical *through-the-cycle* transition matrix, of which the generic element a_{ji} represents the annual probability that an obligor of the credit risk class j will pass to a credit risk class i in the following year. The matrix has dimension $n \times n$ and the elements of row j, $a_{j1}, ..., a_{jn}$ must sum to unity, since every obligor with rating j will certainly be assigned to some credit risk class $z \in \{1, ..., j, ...n\}$ in the year, including the case of being reassigned to the same class j. As a convention, the rows and the columns of the matrix are ordered according to safety class, from the safest (conventionally labeled AAA) to the default (label D: default state).

The *through-the-cycle* transition matrix TTC was estimated using publicly available data[1] of rating grades assigned to sovereign debts by Credit Rating Agencies in the period 1993-2015. This period has been chosen to include aspects of major institutional changes (e.g. events such as the introduction of the euro or the euro zone sovereign debt crisis).The TTC matrix has been estimated by averaging all the available point-in-time transition matrices element-wise. Being the point-in-time transition matrices right stochastic, i.e. real square matrices with each row summing to 1, it is straightforward to show that the resulting TTC matrix is still a right stochastic matrix and it models how each class of credit moves on average (i.e. in the absence of any specific market cycle) to the next credit classes. As a consequence, its decomposition has eigenvalues ≤ 1 with $\max(\lambda_j) = 1$.

Our estimated TTC matrix is reported in Table 1.

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	0.96	0.04	0	0	0	0	0	0
AA	0.02	0.91	0.06	0.01	0	0	0	0
A	0	0.03	0.9	0.07	0	0	0	0
BBB	0	0	0.05	0.87	0.06	0.01	0	0
BB	0	0	0	0.05	0.85	0.08	0.01	0.01
B	0	0	0	0	0.07	0.89	0.03	0.01
CCC	0	0	0	0	0	0.38	0.42	0.19
D	0	0	0	0	0	0	0	1

Table 1: Estimated TTC transition matrix

Given the estimated transition matrix TTC, we apply the following standard diagonalization method for square matrices, by assuming that the it can be decomposed in a Q matrix and a Λ diagonal matrix so that:

$$TTC = Q\Lambda Q^{-1} \tag{1}$$

Since the decomposition is unique unless linear transformations, then Q represents the eigenvectors matrix of the above linear transformation.

Following this model, the **expected cumulative default probability** in the interval $[t, t+\tau]$ is the linear operator given by:

[1] Standard & Poor's Sovereigns Ratings have been downloaded from Bloomberg using a query with parameters:
- RTG_SP_LT_LC_ISSUER_CREDIT
- RATING_AS_OF_DATE_OVERRIDE
- Sovereign Issuer Ticker.

$$\begin{aligned}\mathbf{cdp}(t,t+1) &= Q\Lambda^1 Q^{-1}\mathbf{v}\\ \mathbf{cdp}(t,t+2) &= Q\Lambda^2 Q^{-1}\mathbf{v}\\ \mathbf{cdp}(t,t+\tau) &= Q\Lambda^\tau Q^{-1}\mathbf{v} \quad for \quad \tau > 2\end{aligned} \tag{2}$$

where $\mathbf{cdp}(t,t+T)$, with $T \in [t,\tau]$, is an n-components stochastic process, the j-th element of which, $cdp_{\mathbf{j}}(t,t+T)$, represents the cumulative default probability that an obligor of rating grade class $j = 1,...,n$ will have defaulted by time $t+T$, with $\mathbf{cdp}(t,t) = 0$ and $\mathbf{v}$ a null vector apart its last element equal to 1.

Given the process $\mathbf{cdp}((t,t+\tau))$ in equations (2), the survival probability in the interval $\tau \in [t,t+\tau]$ of an obligor not in default is:

$$\mathbf{sp}(t,t+\tau) = [\mathbf{1} - \mathbf{cdp}(t,t+\tau)] \quad for \quad \tau > 1 \tag{3}$$

where $\mathbf{1}$ is the unit vector.

The expected present value of a vector of unitary annuity maturing at time $t+\tau$ can be written as:

$$\mathbf{a}(t,t+\tau) = \sum_{j=1,...\tau} \frac{1}{(1+r^B)^j}\mathbf{sp}(t,t+j) \tag{4}$$

where r^B represents a common appropriate financial discount rate[2].

Note that the components of vector $\mathbf{a}(t,t+\tau)$ are ordered decreasingly, with the highest rating grades corresponding to higher annuity values since the present value of a unitary annuity is proportional to the survival probability of the corresponding credit risk class and a null value for the vector's last component. Using the eq. (3) and eq. (2), we can rewrite eq. (4) as follows:

$$\mathbf{a}(t,t+\tau) = \sum_{j=1,...\tau} \frac{1}{(1+r^B)^j}\left(\mathbf{1} - Q\Lambda^j Q^{-1}\mathbf{v}\right) \tag{5}$$

By letting $\alpha = 1/(1+r^B)$ and $\beta_j = \lambda_j/(1+r^B)$, with λ_j the j-th eigenvalue in matrix Λ, the above expression can be written:

$$\mathbf{a}(t,t+\tau) = \alpha\frac{1-\alpha^\tau}{1-\alpha}\mathbf{1} - QBQ^{-1}\mathbf{v} \tag{6}$$

where B is a diagonal matrix whose j-th element is $b_j = \beta_j\frac{1-\beta_j^\tau}{1-\beta_j}$. Since the terms in Λ are $\lambda_j \leq 1$, it follows $\alpha, \beta_j \in (0,1)$. By taking the limit for $\tau \to \infty$ we obtain the following perpetual annuity formula:

$$\mathbf{a}(t) = \lim_{\tau\to\infty}\mathbf{a}(t,t+\tau) = \frac{\alpha}{1-\alpha}\mathbf{1} - QB'Q^{-1}\mathbf{v} \tag{7}$$

with B' a diagonal matrix with j-th element $b'_j = \frac{\beta_j}{1-\beta_j}$. The vector $\mathbf{a}(t)$ in the eq. (7) represents expected present values at t of **an annuity of an irredeemable mortgage** paid out by each obligor according to its rating grade.

In order to consider the possibility to recover part of the credit in case an obligor defaults, we should adjust the value of $\mathbf{a}(t)$ accordingly. To this end, eq. (7) should be modified to take into account this effect, by introducing the loss-given-default (LGD), $(1-rr)$, with rr the corresponding recovery rate.[3] The vector of expected values of the recovery rates, for the different rating grades, by time τ is:

$$\mathbf{r}(t,t+\tau) = rr\sum_{j=1,...\tau}\frac{1}{(1+r^B)^j}[\mathbf{cdp}(j) - \mathbf{cdp}(j-1)] \tag{8}$$

where the summation considers the probabilities that a failure occurs at each year j in $[t,t+\tau]$. A closed form for the **expected recovery rate** can be obtained by taking the limit of matrix Λ^τ for $\tau \to \infty$. Substituting for $\mathbf{cdp}()$ and recalling that Λ is a diagonal matrix of constant terms we obtain:

[2]For simplicity's sake, it has been assumed that the purely financial rate does not exhibit a term structure. This hypothesis represents a mere simplification for calculation purposes which can easily be removed.

[3]The LGD parameter should be identified for each Member State in order to take into account its specific risk . Since our ultimate purpose is to provide an exemplification of a possible estimation of the fundamental risk under an irreedimable cost approach, in our calculations we assume a uniform LGD value for all MSs.

$$\begin{aligned}
\mathbf{r}(t,t+\tau) &= rr\left[\sum_{j=1,\ldots\tau}\frac{1}{(1+r^B)^j}Q\Lambda^jQ^{-1} - \frac{1}{(1+r^B)^j}Q\Lambda^j\Lambda^{-1}Q^{-1}\right]\mathbf{v} \\
&= rrQ\left[\sum_{j=1,\ldots\tau}\frac{1}{(1+r^B)^j}\Lambda^j\left(I-\Lambda^{-1}\right)\right]Q^{-1}\mathbf{v} \\
&= rrQ\left[(I-\Lambda^{-1})\sum_{j=1,\ldots\tau}\frac{1}{(1+r^B)^j}\Lambda^j\right]Q^{-1}\mathbf{v}
\end{aligned}$$

By taking the limit for $\tau \to \infty$ we get the following formula for the expected recovery rate:

$$\mathbf{r}(t) = rrQ\left[(I-\Lambda^{-1})B'\right]Q^{-1}\mathbf{v} \tag{9}$$

Following a unitary-payment perpetual amortizing scheme and allowing for partial recovery of funds in case of default, the value of an **expected positive exposure** $\tilde{a}_j$ must always satisfy the equivalence $\tilde{a}_j(1-r_j) = a_j$, where $r_j < 1$ is j-th element of the vector $\mathbf{r}_t$. Taking this in mind, the final expectation of a **unitary perpetual annuity value** at time t calculated for each obligor according to its rating grade j is then:

$$\tilde{a}_j = \frac{a_j}{1-r_j} \tag{10}$$

The vector $\tilde{\mathbf{a}}(t)$, whose elements are the values $\tilde{a}_j$, can be interpreted as a set of perpetual annuities based on **fundamental risk metrics inherent to obligors labelled with specific rating grades**.In our numerical exercise, we set $rr = 0.3$ in the baseline simulation.

Given the above illustrated unitary perpetual annuity measure, the IEP-TPI tool use it to identify the oscillation range within which the yield of a given MS's public debt issuance with rating j should vary according to country's fundamentals. To this end, the tool identifies respectively an upper and a lower bound as follows:

$$\begin{aligned}
\mathbf{UpperBound(j)} &= \frac{1}{\tilde{a}_{j+1}} \\
\mathbf{LowerBound(j)} &= \frac{1}{\tilde{a}_{j-1}}
\end{aligned} \tag{11}$$

For the highest rating the lower band is determined using the same rating grade j.